

ADM/213/438

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ACSIL/ADM/47/979

ARL/RI/E540

ADMIRALTY RESEARCH LABORATORY
TEDDINGTON, MIDDLESEX.

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A SLIDE RULE FOR
RADIATION CALCULATIONS

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SEPTEMBER 1947

ADM/213/438

ADMIRALTY RESEARCH LABORATORY, TEDDINGTON, MIDDLESEX.

A.R.L./R.1/E.540.

A SLIDE RULE FOR RADIATION CALCULATIONS.

SUMMARY.

A slide rule has been developed to facilitate rapid calculations based on the Planck Radiation formula. Quantities such as the radiant flux density in a given wavelength region, or the spectral photon flux density at a given wavelength, can be obtained readily for a black body over a range $\lambda T = 2 \times 10^2$ to $\lambda T = 4 \times 10^6$ micron degrees with an accuracy of about 1%. Simple extension rules can be used for larger values of λT .

I. INTRODUCTION.

In the course of the calculations involved in thermal detection problems it was found that the available tables based on the Planck radiation formula were neither sufficiently comprehensive nor convenient for frequent reference. Accordingly, it was decided to develop a nomograph capable of giving rapidly approximate values for certain of the radiation quantities most frequently encountered. Typical of such quantities is the radiation per unit area emitted by a black body at a given temperature in a given wavelength interval. The general features of the nomograph are shown in Figures 1 and 2.

Simultaneously with the completion of this nomograph, there was received at A.R.L. a simple 8" slide rule, made by Professor Czerny of Frankfurt University, that enabled the radiant flux density in a given wavelength interval for a black body at a given temperature to be calculated roughly. The advantage of such a device was immediately apparent, and it was decided to extend the computations already carried out with a view to the development of a radiation slide rule. The work, involving some 1000 man hours of computing, has now been completed and prototype slide rules made. This note gives a brief description of the slide rule and the methods used in the associated calculations. Instructions for use, and examples of the solution of typical problems are included, in order that this note may serve as an instruction booklet for any future slide rules.

II. DESCRIPTION OF THE SLIDE RULE.

In the final form envisaged the slide rule will include the scales required for calculations involving spectral radiant flux density and radiant flux density, together with the corresponding quantities in photon units. The three prototype rules so far constructed contained scales enabling immediate calculation to be made of a greater number of quantities, but as a result of experience with these prototypes the number of scales on the final version will be reduced. The scales of the original rules were reproduced photographically from

a double full size rule fitted with hand drawn scales. The size of the latter rule was $39\frac{1}{2}" \times 10\frac{1}{4}"$, of the prototype, $19\frac{3}{4}" \times 5"$, while future models will be $19\frac{3}{4}" \times 3\frac{1}{2}"$.

It is proposed that the stock of the rule shall carry scales of maximum spectral radiant flux density, $H_{\lambda \text{ max}}$, total radiant flux density, H , maximum spectral photon flux density, $Q_{\lambda \text{ max}}$, total photon flux density, Q , temperature (Centigrade and Absolute), and wavelength. One side of the slide will carry the scales required for calculations involving the first two of the above quantities, while the scales necessary for photon calculations will be on the reverse of the slide. The back of the stock will carry brief definitions and instructions, together with scales relating wavelength, wave-number, and electron volts.

III. DEFINITIONS, SYMBOLS AND UNITS.

The subscripts $\lambda, \Delta\lambda, \nu, \Delta\nu$, refer to wavelength, wavelength interval, wave-number, and wave-number interval respectively. The symbol cm will be used to denote unit of length referring to the surface of a black body.

λ : wavelength in cm_{λ}

ν : wave-number in cm_{ν}^{-1}

t : temperature in degrees Centigrade

T : temperature in degrees Absolute ($= t + 273.18$)

c : velocity of light ($= 2.99776 \times 10^{10} \text{ cm sec}^{-1}$)

k : Boltzmann constant ($= 1.38047 \times 10^{-16} \text{ erg T}^{-1}$)

h : Planck constant ($= 6.6242 \times 10^{-27} \text{ erg sec}$)

$c_1 = 2\pi h c^2$ ($= 3.7403 \times 10^8 \text{ watt cm}^2$)

$c_2 = hc/k$ ($= 1.43848 \text{ cm T}$)

$c'_1 = 2\pi c$ ($= 1.88355 \times 10^{11} \text{ cm sec}^{-1}$)

$$x = \frac{c_2}{\lambda T} \quad dx = - \frac{c_2 d\lambda}{\lambda^2 T}$$

The values of the constants c , k and h are taken from Birge, Rev. of Mod. Phys. 13,233, 1941.

H : Total Radiant flux density (watt cm^{-2})

$H_{\lambda_1-\lambda_2}$: Radiant flux density (watt cm^{-2})

H_{λ} : Spectral radiant flux density ($\text{watt cm}^{-2} \text{ cm}_{\Delta\lambda}^{-1}$)

Q : Total photon flux density ($\text{photon sec}^{-1} \text{ cm}^{-2}$)

$Q_{\lambda_1-\lambda_2}$: Photon flux density ($\text{photon sec}^{-1} \text{ cm}^{-2}$)

Q_{λ} : Spectral photon flux density ($\text{photon sec}^{-1} \text{ cm}^{-2} \text{ cm}_{\Delta\lambda}^{-1}$)

IV./

IV. THE PLANCK RADIATION FORMULA.

The Planck radiation formula may be written:

$$H_{\lambda} = \frac{c_1 \lambda^{-5}}{e^{c_2/\lambda T} - 1} \text{ watt } \frac{\text{cm}}{\text{cm}^2} \frac{-2}{\text{cm}_{\Delta\lambda}^{-1}},$$

where H_{λ} is the spectral radiant flux density, that is, the energy radiated in all directions by unit area of a black body at temperature T at wavelength λ and for a wavelength interval $\Delta\lambda = 1 \text{ cm}_{\Delta\lambda}^*$. The corresponding expression for wave-number interval $\Delta\nu = 1 \text{ cm}_{\Delta\nu}^{-1}$ is:

$$H_{\nu} = \frac{c_1 \lambda^{-3}}{e^{c_2/\lambda T} - 1} \text{ watt } \frac{\text{cm}}{\text{cm}^2} \frac{-2}{\text{cm}_{\Delta\nu}^{-1}},$$

which form is sometimes more convenient.

Since the energy associated with one photon is hc/λ , the corresponding spectral photon flux density is given by:

$$Q_{\lambda} = \frac{c_1 \lambda^{-4}}{e^{c_2/\lambda T} - 1} \text{ photon sec } \frac{\text{cm}}{\text{cm}^2} \frac{-1}{\text{cm}_{\Delta\lambda}^{-1}}.$$

For a wavelength interval $d\lambda$ the energy radiated is $H_{\lambda}d\lambda$, and for a wavelength interval between λ_1 and λ_2 :

$$H_{\lambda_1-\lambda_2} = \int_{\lambda_1}^{\lambda_2} H_{\lambda} d\lambda = \int_0^{\lambda_2} H_{\lambda} d\lambda - \int_0^{\lambda_1} H_{\lambda} d\lambda = \int_{\lambda_1}^{\infty} H_{\lambda} d\lambda - \int_{\lambda_2}^{\infty} H_{\lambda} d\lambda.$$

For a given temperature T , the wavelength λ_m for which H_{λ} is a maximum, is obtained from $dH/d\lambda = 0$, which gives, of course, one case of Wien's Displacement Law:

$$\lambda_m = \frac{2897.2}{T} \text{ microns.}$$

By substitution:

$$H_{\lambda\text{max}} = 1.2875 \times 10^{-11} T^5 \text{ watt } \frac{\text{cm}}{\text{cm}^2} \frac{-2}{\text{cm}_{\Delta\lambda}^{-1}}.$$

The total radiant flux density H , is obtained by integrating H_{λ} over the wavelength range from zero to infinity, which process leads to the Stefan-Boltzmann Law of Radiation:

$$H = H_{0-\infty} = 5.6728 \times 10^{-12} T^4 = \sigma T^4 \text{ watt } \frac{\text{cm}}{\text{cm}^2} \frac{-2}{\text{cm}_{\Delta\lambda}^{-1}}.$$

The/

* It is not possible to attach physical significance to the use of these wavelength and wave-number intervals. The spectral quantities refer to a single point on a radiation curve and in all calculations involving real wavelength interval, the centimetre interval disappears.

The corresponding formulae may be obtained in terms of photons:

$$Q_{\lambda_1-\lambda_2} = \int_{\lambda_1}^{\lambda_2} Q_{\lambda} d\lambda = \int_0^{\lambda_2} Q_{\lambda} d\lambda - \int_0^{\lambda_1} Q_{\lambda} d\lambda = \int_{\lambda_1}^{\infty} Q_{\lambda} d\lambda - \int_{\lambda_2}^{\infty} Q_{\lambda} d\lambda,$$

$$Q_{\nu} = \frac{c_1 \lambda^{-2}}{e^{c_2/\lambda T} - 1} \text{ photon sec}^{-1} \text{ cm}^{-2} \text{ cm}_{\Delta\lambda},$$

$$\lambda'_m = \frac{3668.9}{T} \text{ microns},$$

$$Q_{\lambda \text{ max}} = 2.1028 \times 10^{11} T^4 \text{ photon sec}^{-1} \text{ cm}^{-2} \text{ cm}_{\Delta\lambda},$$

$$Q = Q_{0-\infty} = 1.5124 \times 10^{11} T^3 = \sigma' T^3 \text{ photon sec}^{-1} \text{ cm}^{-2}.$$

To the various formulae given above, Wien's Displacement Law may be applied.

This states that for a series of black body radiation curves for temperatures $T_1, T_2, \dots, T_n$, points may be chosen on the different curves corresponding to wavelengths $\lambda_1, \lambda_2, \dots, \lambda_n$ such that the product λT is constant:

$$\lambda_1 T_1 = \lambda_2 T_2 = \dots = \lambda_n T_n = \text{Constant}$$

Thus any quantity known for temperature T_1 may be derived for temperature T_2 using the factor $(T_1/T_2)^n$ where n is a constant for any given formula.

V. EXISTING TABLES.

The published tables (e.g., 5, 6, 7 & 8) are based on the Wien Displacement Law. Thus for the evaluation of H_{λ} for temperature T_g and wavelength λ_g two forms of table are available.

(1) Tables of $H_{\lambda} = f(\lambda)$ for some temperature T_t (usually 1000°K or 10000°K).

λ_g is replaced by $\lambda_t = \lambda_g (T_g/T_t)$, and the appropriate value of $(H_{\lambda})_t$ found from the tables, when:

$$(H_{\lambda})_g = (H_{\lambda})_t \times (T_g/T_t)^5.$$

(2) Tables of $H_{\lambda}/H_{\lambda \text{ max}} = f(\lambda T)$, the values of which are independent of temperature.

The value of the fraction is found for $\lambda_g T_g$, and since $H_{\lambda \text{ max}}$ can be calculated $(=\sigma T^5)$, H_{λ} is obtained.

Tables of the first type are available for $H_{0-\lambda}$, Q_{λ} and $Q_{0-\lambda}$, and of the second type for $H_{0-\lambda}/H$, $Q_{\lambda}/Q_{\lambda \text{ max}}$, and $Q_{0-\lambda}/Q$.

These tables give adequate accuracy but involve additional calculations.

Moore has formed an approximation to the Planck formula for the visible region by replacing it by a polynomial of the second degree:

$$H = k_0 + k_1 \lambda + k_2 \lambda^2$$

for certain values of temperature. In the visible wavelength interval the polynomial coincides with the Planck formula for 3 values of wavelength. Variation of the coefficients

$$k_0, k_1/$$

k_0, k_1 and k_2 with temperature is, however, rapid, and application of this method is not easy.

For shorter or longer wavelengths the simple Wien or Rayleigh-Jeans formulae may be applied. These are approximations only, and it is necessary to have a method of determining the error introduced by their use. This error will, in general, be dependent on the range of λT involved, and a method of estimating it will be described later.

The processes involved in the use of the slide rule are described below. The exact Planckian formula may be applied over the whole range of temperature and over an unlimited range of wavelength in the direction of longer wavelengths.

VI. GENERAL METHOD USED IN CALCULATIONS.

The various formulae given in the preceding section may be written:

$$H_\lambda = \frac{c_1 T^5 x^5 X}{c_2^5}, \quad H_\nu = \frac{c_1 T^3 x^3 X}{c_2^3}, \quad H_{\lambda_1 - \lambda_2} = \frac{c_1 T^4}{c_2^4} \int_{x_1}^{x_2} -x^3 X dx,$$

$$Q_\lambda = \frac{c_1' T^4 x^4 X}{c_2^4}, \quad Q_\nu = \frac{c_1' T^2 x^2 X}{c_2^2}, \quad Q_{\lambda_1 - \lambda_2} = \frac{c_1' T^3}{c_2^3} \int_{x_1}^{x_2} -x^2 X dx,$$

where $x = c_2/\lambda T$ and for simplicity, $X = (e^x - 1)^{-1}$.

These may be reduced to four basic formulae which are sufficient for all calculations involving black body radiation:

$$H_\lambda = c_1 \left(\frac{T}{c_2} \right)^5 x^5 X$$

$$H_{0-\lambda} = c_1 \left(\frac{T}{c_2} \right)^4 \int_{\infty}^x -x^3 X dx$$

$$Q_\lambda = c_1' \left(\frac{T}{c_2} \right)^4 x^4 X$$

$$Q_{0-\lambda} = c_1' \left(\frac{T}{c_2} \right)^3 \int_{\infty}^x -x^2 X dx$$

of which the general form is:

$$BT^{-n} = \text{Const. } f(x) \quad n = 3, 4, 5,$$

where B is any one of the basic quantities, H_λ , Q_λ , etc.

BT^{-n} is a function of x only, and for each value of x there will be a sequence of values of λ related to the corresponding values of T by the formula $\lambda T = c_2/x = \text{constant}$. To find the value of B for given values of λ and T , it is necessary to calculate $x = c_2/\lambda T$ and hence BT^{-n} and, finally,

$$(BT^{-n})T^n = B.$$

If the radiating body obeys Lambert's Law, quantities associated with the radiation in unit solid angle normal to the radiating surface may be obtained by dividing the above formulae by π . To avoid multiplicity of scales such quantities have not been included on the slide rule.

VII. TRANSFORMATIONS OF FORMULAE INVOLVED IN CALCULATION OF SCALES.

(a) Spectral formulae:

A general formula for H_λ , H_ν , Q_λ and Q_ν may be written:

$$B = \text{Const. } T^n \cdot \frac{x^n}{e^x - 1} \quad (n = 3, 4, 5).$$

The physical limits of x are zero and infinity; B is always positive and when x tends to zero or infinity, B tends to zero. If the maximum value of B given by $dB/dx = 0$ corresponds to $x = \beta$, then:

$$e^{-\beta} + \frac{\beta}{n} - 1 = 0.$$

The quantities associated with the maxima are given in Table I.

TABLE I.

Quantity	H_λ	Q_λ	H_ν	Q_ν
n	5	4	3	2
β	4.965114	3.92069	2.82144	1.59362
λ_m (symbol)	λ_m	λ'_m	λ''_m	λ'''_m
$\lambda_m T = c_2/\beta$ ($\mu^\circ K$)	2897.2	3668.9	5098.4	9026.5
Max of $\frac{x^n}{e^x - 1}$	$c_4 = 21.2016$	$c'_4 = 4.77984$	$c''_4 = 1.42144$	$c'''_4 = 0.647611$
$H_{\lambda \text{ max, etc.}}$	$1.2875 \times 10^{-11} T^5$	$2.1027 \times 10^{-11} T^4$	$1.7862 \times 10^{-12} T^3$	$1.52132 \times 10^{-11} T^2$

The ratios of the various quantities to their maximum values are given by:

$$\frac{H_\lambda}{H_{\lambda \text{ max}}} = \frac{1}{c_4} \cdot \frac{x^5}{e^x - 1}, \quad \frac{Q_\lambda}{Q_{\lambda \text{ max}}} = \frac{1}{c'_4} \cdot \frac{x^4}{e^x - 1}$$

and it is sufficient to know these two ratios in order to evaluate H_λ , H_ν , Q_λ and Q_ν , since $H_\nu = (H_\lambda / H_{\lambda \text{ max}}) \times H_{\lambda \text{ max}} \times \lambda_\mu^2 \times 10^{-8}$ and $Q_\nu = (Q_\lambda / Q_{\lambda \text{ max}}) \times Q_{\lambda \text{ max}} \times \lambda_\mu^2 \times 10^{-8}$, where λ is expressed in microns.

(b) Integration formulae:

$$H_{\lambda_1 - \lambda_2} = H_{\nu_1 - \nu_2} = \int_{\lambda_1}^{\lambda_2} H_\lambda d\lambda = \int_{\nu_1}^{\nu_2} H_\nu d\nu = c_1 \left(\frac{T}{c_2} \right)^4 \int_{x_1}^{x_2} \frac{x^3 dx}{e^x - 1},$$

where $x_1 = c_2/\lambda_1 T = c_2 \nu_1 / T$, $x_2 = c_2/\lambda_2 T = c_2 \nu_2 / T$, and $dx = -\frac{c_2 d\lambda}{\lambda^2 T}$.

Expanding $(e^x - 1)$ this becomes:

$$H_{\lambda_1 - \lambda_2} = c_1 \left(\frac{T}{c_2} \right)^4 \int_{x_1}^{x_2} - (e^{-x} + e^{-2x} + \dots) x^3 dx,$$

which on integration by parts gives:

$$H_{\lambda_1 - \lambda_2} = c_1 \left(\frac{T}{c_2} \right)^4 \left[\sum_{n=1}^{\infty} \frac{1}{n^4} [(nx)^3 + 3 (nx)^2 + 6 nx + 6] e^{-nx} \right]_{x_1}^{x_2}.$$

For/

For $\lambda_1 = 0, x_1 \rightarrow \infty$ and

$$H_{0-\lambda} = c_1 \left(\frac{T}{c_2} \right)^4 \sum_{n=1}^{\infty} \frac{1}{n^4} [(nx)^3 + 3(nx)^2 + 6nx + 6] e^{-nx}.$$

Also for $\lambda_2 \rightarrow \infty, x_2 = 0$ and

$$H = H_{0-\infty} = c_1 \left(\frac{T}{c_2} \right)^4 \times 6 \times (1^{-4} + 2^{-4} + 3^{-4} + \dots).$$

$$\text{Since } \sum_{n=1}^{\infty} n^{-4} = \frac{\pi^4}{90},$$

$$H = c_1 \left(\frac{T}{c_2} \right)^4 \frac{\pi^4}{15} = \sigma T^4 = 5.6728 \times 10^{-12} T^4,$$

which is the expression for the total power radiated per unit area of a black body at temperature T.

$$\text{Also } \frac{H_{0-\lambda}}{H} = \frac{15}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} [(nx)^3 + 3(nx)^2 + 6nx + 6] e^{-nx}$$

For large values of $\lambda : H_{0-\lambda} \approx H$. In such cases it is better to use $H_{\lambda-\infty} = H - H_{0-\lambda}$.

$$\text{Thus } \frac{H_{\lambda-\infty}}{H} = 1 - \frac{H_{0-\lambda}}{H} \text{ or } \frac{H_{\lambda-\infty}}{H} + \frac{H_{0-\lambda}}{H} = 1.$$

Since $x = c_2/\lambda T$, the above formulae can be expressed as two general functions

$$\frac{H_{0-\lambda}}{H} = f_1(\lambda T) \text{ and } \frac{H_{\lambda-\infty}}{H} = f_2(\lambda T),$$

the graphs of which are given in Figure 2.

Similarly for $Q_{\lambda_1-\lambda_2}$:

$$Q_{0-\lambda} = c_1' \left(\frac{T}{c_2} \right)^3 \sum_{n=1}^{\infty} \frac{1}{n^3} [(nx)^2 + 2nx + 2] e^{-nx}$$

$$\text{and } Q = Q_{0-\infty} = c_1' \left(\frac{T}{c_2} \right)^3 \times 2 \times \frac{\pi^3}{25.79436 \dots} = \sigma' T^3 = 1.5214 \times 10^{11} T^3.$$

For large values of λ , $Q_{0-\lambda}$ is replaced by $Q_{\lambda-\infty} = Q - Q_{0-\lambda}$ when

$$\frac{Q_{0-\lambda}}{Q} = 1 - \frac{Q_{\lambda-\infty}}{Q} \text{ or } \frac{Q_{0-\lambda}}{Q} + \frac{Q_{\lambda-\infty}}{Q} = 1.$$

In Figure 2 the two curves $Q_{0-\lambda}/Q$ and $Q_{\lambda-\infty}/Q$ are also shown.

VIII. THE SCALES OF THE SLIDE RULE.

Stock:

Taking as a basis the value of T, the formulae listed below are obtained by taking logarithms of both sides of the formulae already given:

$$\log \lambda_m = \log \frac{c_2}{\sigma} - \log T$$

$$\log H_{\lambda \max} = \log \frac{c_1 c_4}{c_2^5} + 5 \log T$$

$$\log H = \log \sigma + 4 \log T$$

$$\log Q_{\lambda \max} = \log \frac{c_1' c_4'}{c_2^4} + 4 \log T$$

$$\log Q = \log \sigma' + 3 \log T$$

If/

If T is plotted as a logarithmic scale (d) with modulus m_T , the remaining scales are also logarithmic with moduli:

$$m_\lambda = -m_T \text{ (inverse scale e)}$$

$$m_{H_\lambda \max} = \frac{1}{5} m_T \text{ (scale b)}$$

$$m_H = \frac{1}{4} m_T \text{ (scale a)}$$

$$m_{Q_\lambda \max} = \frac{1}{4} m_T \text{ (scale g)}$$

$$m_Q = \frac{1}{3} m_T \text{ (scale f)}$$

The constants in the various equations define the positions of the corresponding scales relative to the temperature scale. These constants are of two types, firstly those of purely mathematical conception and so invariant (e.g., β , c_4 , c_4'), and secondly those derived from physical constants (c_2 , c_1 , c_1') which will depend on the values taken for c , h , and k . The values of these constants used have already been listed. The procedure necessary for use of the slide rule with changed values of the constants is given in the next section.

Since in practical problems temperatures are usually expressed in degrees Centigrade, the appropriate scale has been included (c). It has been assumed that $0^\circ\text{C} = 273.18^\circ\text{K}$.

In the construction of the slide rule, a modulus for the temperature scale has been chosen $m_T = 20$ cm, and about two decades taken covering directly the range 100°K to $10,000^\circ\text{K}$ (-180°C to about $10,000^\circ\text{C}$). The same scale of wavelength is used to cover two ranges 0.3 to 30 microns (black figures) and 30 to 3000 microns (3 mm) (red figures).

All the scales on the stock, with the exception of scale e are purely logarithmic, differing only by the moduli. In deriving the positions of the scale lines, involved fractions were calculated for round values of argument x , and then, with the aid of Interpolation and Allied Tables (HMSO, 1936), were inversely interpolated for round values of fractions using Bessel's interpolation formula including fourth and sometimes sixth differences. All calculations were made with two Brunsviga calculating machines by the method of balancing approximation.

Slide:

The slide is double sided, one side (energy) for use in the H , H_λ , and H_ν calculations, the other (photon) for Q , Q_λ , and Q_ν . Since the two sides are similar only one will be discussed in detail.

The scales of the slide are independent of the stock except in so far as the moduli are determined by m_T . The two upper scales are those of the fraction $H_\lambda/H_{\lambda \max} = \frac{1}{c_4} \cdot \frac{x^5}{e^x - 1}$ for the two wavelength ranges. Since the only constant here, c_4 , is independent of the physical constants, the scale itself is independent.

For/

For $x = \beta$, $H_\lambda/H_{\lambda\max} = 1$ and at this point a line marked " $H_{\lambda\max}$ " is added to the slide. If the slide is moved so that this line is opposite a given temperature, T , then the upper end of the line is opposite the corresponding value of λ_m on scale e. Scale h or i gives the values of $H_\lambda/H_{\lambda\max}$ corresponding to any other value of λ . Four lines have been added to scale h giving the wavelengths corresponding to maximum values of H_λ , H_ν , ψ_λ , and Q_ν .

If scale e is extended to longer wavelengths by two decades, the corresponding values of scale h go down to 10^{-11} . The extended wavelength scale may be displaced to coincide with the original scale with values of λ 100 times larger as given by the red figures. The corresponding scale of $H_\lambda/H_{\lambda\max}$ (i) is also coloured red. The fact that, in the direction of longer wavelengths, scale i approximates to a logarithmic scale of modulus $\frac{1}{4} m_T = \frac{1}{4} m_\lambda$, permits the extension of the rule in this direction.

In the same way scale m (black) covers the shorter wavelength range:

$$\frac{H_{\infty-\lambda}}{H} = \frac{15}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} [(nx)^3 + 3(nx)^2 + 6nx + 6] e^{-nx},$$

and scale l (red), the longer range:

$$\frac{H_{\lambda-\infty}}{H} = 1 - \frac{15}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} [(nx)^3 + 3(nx)^2 + 6nx + 6] e^{-nx}.$$

Scale l approximates to a logarithmic scale of modulus $\frac{1}{3} m_T$

On the photon side of the slide are the corresponding scales of $Q_\lambda/Q_{\lambda\max} = \frac{1}{c_4} \cdot \frac{x^4}{e^x - 1}$ (n, o), and

$$\frac{Q_{\infty-\lambda}}{Q} = \frac{12.89718}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} [(nx)^2 + 2(nx) + 2] e^{-nx} \quad (s),$$

$$\frac{Q_{\lambda-\infty}}{Q} = 1 - \frac{12.89718}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} [(nx)^2 + 2nx + 2] e^{-nx} \quad (r).$$

Scales o and r approximate to logarithmic scales of moduli $\frac{1}{3} m_T$ and $\frac{1}{2} m_T$ respectively.

For convenience in certain calculations, two logarithmic scales are included on each side of the slide, j and k on the energy side with moduli $\frac{1}{5} m_T$ and $\frac{1}{4} m_T$, and p and q on the photon side with moduli $\frac{1}{4} m_T$ and $\frac{1}{3} m_T$.

Back of stock:

Scales relating wavelength, wave-number and electron volts are included on the back of the stock. For convenience in reading from these scales without use of a cursor the wave-number scale is given twice.

IX. LIST OF SCALES./

IX. LIST OF SCALES.

Stock:

- a Total radiant flux density, H watt cm^{-2}
- b Maximum spectral radiant flux density, $H_{\lambda\text{max}}$ watt $\text{cm}^{-2} \text{cm}^{-1}_{\Delta\lambda}$
- c Temperature, $t^{\circ}\text{C}$.
- d Temperature, $T^{\circ}\text{K}$.
- e Wavelength, λ microns: black from 0.3 to 30; red from 30 to 3000
- f Total photon flux density, Q photon $\text{sec}^{-1} \text{cm}^{-2}$
- g Maximum spectral photon flux density, $Q_{\lambda\text{max}}$ photon $\text{sec}^{-1} \text{cm}^{-2} \text{cm}^{-1}_{\Delta\lambda}$

Slide - energy side:

- h Fraction $H_{\lambda}/H_{\lambda\text{max}}$ (for $\lambda = 0.3 - 30$ microns)
- i Fraction $H_{\lambda}/H_{\lambda\text{max}}$ (for $\lambda = 30 - 3000$ microns-red values)
- j Multiplication scale for use with b ($H_{\lambda\text{max}}$)
- k Multiplication scale for use with a (H)
- l Fraction $H_{\lambda-\infty}/H$ (for $\lambda = 30 - 3000$ microns - red values)
- m Fraction $H_{0-\lambda}/H$ (for $\lambda = 0.3 - 30$ microns)

Slide - photon side:

- n Fraction $Q_{\lambda}/Q_{\lambda\text{max}}$ (for $\lambda = 0.3 - 30$ microns)
- o Fraction $Q_{\lambda}/Q_{\lambda\text{max}}$ (for $\lambda = 30 - 3000$ microns - red values)
- p Multiplication scale for use with g ($Q_{\lambda\text{max}}$)
- q Multiplication scale for use with f (Q)
- r Fraction $Q_{\lambda-\infty}/Q$ (for $\lambda = 30 - 3000$ microns - red values)
- s Fraction $Q_{0-\lambda}/Q$ (for $\lambda = 0.3 - 30$ microns)

Back of stock:

Wavelength
Wave-number (twice)
Electron volts

X. USE OF THE SLIDE RULE.

(i) If the cursor line is set on a given value of temperature on scale d ($^{\circ}\text{K}$) or c ($^{\circ}\text{C}$), then the following data may be read directly from the cursor line:

- scale e: Wavelength, λ_m , corresponding to maximum radiant flux density, $H_{\lambda\text{max}}$
- scale a: Total radiant flux density, H .
- scale b: Maximum spectral radiant flux density, $H_{\lambda\text{max}}$
- scale f: Total photon flux density, Q .
- scale g: Maximum spectral photon flux density, $Q_{\lambda\text{max}}$

(ii) If the line marked " $H_{\lambda\text{max}}$ " on the energy side of the slide is brought into coincidence with a given temperature, and the cursor line placed against a given wavelength, the values of the following fractions may be read directly from the cursor line:

scale/

scale h: $H_\lambda/H_{\lambda\max}$ (for $\lambda = 0.3 - 30$ microns)
 scale i: $H_\lambda/H_{\lambda\max}$ (for $\lambda = 30 - 3000$ microns)
 scale m: $H_{0-\lambda}/H$ (for $\lambda = 0.3 - 30$ microns)
 scale l: $H_{\lambda-\infty}/H$ (for $\lambda = 30 - 3000$ microns)

If the " $H_{\lambda\max}$ " line on the photon side is used the following fractions are obtained:

scale n: $Q_\lambda/Q_{\lambda\max}$ (for $\lambda = 0.3 - 30$ microns)
 scale o: $Q_\lambda/Q_{\lambda\max}$ (for $\lambda = 30 - 3000$ microns)
 scale s: $Q_{0-\lambda}/Q$ (for $\lambda = 0.3 - 30$ microns)
 scale r: $Q_{\lambda-\infty}/Q$ (for $\lambda = 30 - 3000$ microns)

Further quantities may then be estimated using the formulae below:

$$\begin{aligned} H_\lambda &= (H_\lambda/H_{\lambda\max}) \times H_{\lambda\max} \\ H_\nu &= (H_\lambda/H_{\lambda\max}) \times H_{\lambda\max} \times \lambda_\mu^2 \times 10^{-8} \\ H_{\lambda_1-\lambda_2} &= [(H_{0-\lambda_2}/H) - (H_{0-\lambda_1}/H)] \times H \\ \text{or } H_{\lambda_1-\lambda_2} &= [(H_{\lambda_1-\infty}/H) - (H_{\lambda_2-\infty}/H)] \times H \\ Q_\lambda &= (Q_\lambda/Q_{\lambda\max}) \times Q_{\lambda\max} \\ Q_\nu &= (Q_\lambda/Q_{\lambda\max}) \times Q_{\lambda\max} \times \lambda_\mu^2 \times 10^{-8} \\ Q_{\lambda_1-\lambda_2} &= [(Q_{0-\lambda_2}/Q) - (Q_{0-\lambda_1}/Q)] \times Q \\ \text{or } Q_{\lambda_1-\lambda_2} &= [(Q_{\lambda_1-\infty}/Q) - (Q_{\lambda_2-\infty}/Q)] \times Q \end{aligned}$$

In the formulae with square brackets only positive values of differences are taken. Both values of the fraction inside the bracket must be read from the same scale, or, if this is not possible since one wavelength is within the black range and the other within the red, then one of the fractions must be subtracted from unity.

If the cursor line is placed in turn on the four lines on scales h or n marked $H_{\lambda\max}$, $H_{\nu\max}$, $Q_{\lambda\max}$, $Q_{\nu\max}$, the values of wavelength, λ_m , λ'_m , λ''_m , λ'''_m , for which these quantities have their maximum values at the given temperature (opposite $H_{\lambda\max}$), can be read from scale e.

Two multiplication scales are included on each side of the slide (energy side - j and k; photon side - p and q) to simplify certain calculations. These scales are so arranged that it is unnecessary to move the slide, when set at a given temperature, in order to carry out the multiplication. Care must be taken, however, to use the correct multiplication scale for a given operation:

scale j: with scale b ($H_{\lambda\max}$)
 scale k: with scale a (η)
 scale p: with scale e ($Q_{\lambda\max}$)
 scale q: with scale f (ϵ)

When $H_{\lambda_1-\lambda_2}$ or $Q_{\lambda_1-\lambda_2}$ calculations are being carried out for a narrow wavelength interval, the accuracy is increased by calculating H_λ , H_ν , Q_λ or Q_ν for the middle point of the interval, and, assuming the spectral quantity to be constant over the interval, multiplying the quantity by the width of the interval.

XI. EXTENSION RULES AND CHANGE OF CONSTANTS.

Temperature:

The slide rule may be used for calculations involving temperatures outside the range 100°K to 10,000°K by a simple extension process. The temperature T , is expressed in Degrees Absolute as $T_s = T \times 10^n$, where n is a convenient integer such that T_s is within the compass of scale d . The corresponding wavelength is given by $\lambda_s = \lambda \times 10^{-n}$. By the processes already described, the values of the various quantities B_s for temperature T_s and wavelength λ_s are found, and the values appropriate to the original temperature obtained from the following formulae (given in column VIII on the back of the rule):

$$\begin{aligned} H_\lambda &= (H_\lambda)_s \times 10^{-5n}; & Q_\lambda &= (Q_\lambda)_s \times 10^{-4n} \\ H_\nu &= (H_\nu)_s \times 10^{-3n}; & Q_\nu &= (Q_\nu)_s \times 10^{-2n} \\ H_{\lambda_1-\lambda_2} &= (H_{\lambda_1-\lambda_2})_s \times 10^{-4n}; & Q_{\lambda_1-\lambda_2} &= (Q_{\lambda_1-\lambda_2})_s \times 10^{-3n} \end{aligned}$$

A similar process may be applied to problems in which, although T is inside the normal range, the calculations move outside the limits of the slide.

Longer wavelengths:

Since the red scales covering the longer wavelength range to 3000 microns (i, l, o, r) approximate to logarithmic, the slide rule may be extended to longer wavelengths. The wavelength, is replaced by a wavelength, λ_s , such that $\lambda_s = \lambda \times 10^{-n}$ where n is the smallest possible positive integer that will bring λ_s inside the normal red wavelength range of the rule. The corresponding quantities B_s may be calculated and transposed to the values for the original wavelength using the formulae given for the extension of the temperature scales.

Change of constants:

The values used for the various physical constants have already been listed. If it is desired to introduce new values nc_1 , nc_2 , nc'_1 , it is only necessary to make all calculations for a temperature T_n where $T_n = T \times c_2/nc_2$ and multiply the values so obtained by the following factors:

$$\begin{aligned} \text{Energy} \quad \frac{nc_1}{c_1} &= \frac{h'c'^2}{hc^2} \\ \text{Photons} \quad \frac{nc'_1}{c'_1} &= \frac{c'}{c} \end{aligned}$$

where c' and h' are the new values of the constants. This method is derived from the formulae given at the beginning of section VI, and in column IV on the back of the rule.

XII. ACCURACY.

The accuracy with which calculations can be made is limited by the accuracy with which the scales can be read. For any logarithmic scale the relative error is constant and approximately inversely proportional to the modulus of the scale. In the case of the scales of $H_{\lambda\max}$ (b), H (a), $Q_{\lambda\max}$ (g) and Q (f) (the scales with the smaller moduli) the error can be eliminated if desired by using the exact formulae given in section IV and in column VI on the back of the rule:

$$H_{\lambda\max} = 1.2875 \times 10^{-11} T^5,$$

$$H = 5.6728 \times 10^{-12} T^4,$$

$$Q_{\lambda\max} = 2.107 \times 10^{11} T^4,$$

$$Q = 1.5213 \times 10^{11} T^3.$$

The logarithmic scales influencing the accuracy of any calculation are those of temperature and wavelength. Both scales are of modulus 20 cm. and the accuracy of reading is of the order of 0.25%. The other scales involved are those of the fractions (h, i, l, m, n, o, r, s) which are not logarithmic. In general the relative accuracy will vary according to the region of the scale in use. However, for the longer wavelengths (red scales) the errors are constant, the values being:

$$H_{\lambda}/H_{\lambda\max} (i) : \text{about } 1\%,$$

$$H_{\lambda-\infty}/H \text{ and } Q_{\lambda}/Q_{\lambda\max} (\underline{l} \text{ and } \underline{o}) : \text{about } 0.75\%,$$

$$Q_{\lambda-\infty}/Q (r) : \text{about } 0.5\%.$$

In the region of the respective maxima, the relative error for the fractions is reduced to a few tenths of a per cent, and increases from this value as shown in the table below, being approximately the same for all scales:

	Long wavelengths.	Vicinity of maximum spectral value.	Shorter wavelengths Values of fraction				
			10^{-3}	10^{-6}	10^{-11}	10^{-16}	10^{-25}
$H_{\lambda}/H_{\lambda\max}$	1%						
$H_{\lambda-\infty}/H$ or $H_{\lambda-\lambda}/H$	0.75%						
$Q_{\lambda}/Q_{\lambda\max}$	0.75%	0.1 - 0.5%	2.5%	5%	7.5%	10%	15%
$Q_{\lambda-\infty}/Q$ or $Q_{\lambda-\lambda}/Q$	0.5%						

In $H_{\lambda_1-\lambda_2}$ or $Q_{\lambda_1-\lambda_2}$ calculations the relative error is increased by taking differences, the error being greater for a narrow interval. For this reason it is better to assume a constant value of H_{λ} or Q_{λ} over the interval as described in section X.

XIII. EXAMPLES.

Some typical problems are solved below using the slide rule. Since the calculations have been carried out on one of the prototype rules, the various values quoted may differ in the final decimal place, from values obtained with any future production rules. To demonstrate, in a small number of examples, most of the processes involved in the use of the rule, some of the problems are not directly related to practical conditions.

1. Given a black body at 1650°C find (i) the maximum spectral flux density $H_{\lambda\text{max}}$ and the corresponding wavelength λ_m , (ii) the spectral radiant flux densities for wavelengths 0.5, 3, and 50 microns, and (iii) the wavelength corresponding to $H_{\lambda\text{max}}$ and H_{ν} for the above three wavelengths.

Set the line $H_{\lambda\text{max}}$ on the energy side of the slide against the point $t = 1650^{\circ}\text{C}$ on scale e. Place the cursor in coincidence with the line $H_{\lambda\text{max}}$ and read:

(i) on the scale b: $H_{\lambda\text{max}} = 3.42 \times 10^{-5} \text{ watt cm}^{-2} \text{ cm}_{\Delta\lambda}^{-1}$,

on scale e: $\lambda_m = 1.505 \text{ microns}$;

(ii) leaving the slide in position, place the cursor line on the values 0.5, 3, and 50 microns in turn on the scale e (the value for 50 microns being taken on the red figures) and read the appropriate values of the fraction $H_{\lambda}/H_{\lambda\text{max}}$:

on scale h: for $\lambda = 0.5$; $H_{\lambda}/H_{\lambda\text{max}} = 1.13 \times 10^{-2}$;

for $\lambda = 3$; $H_{\lambda}/H_{\lambda\text{max}} = 0.411$;

on scale i (red): for $\lambda = 50$; $H_{\lambda}/H_{\lambda\text{max}} = 2.19 \times 10^{-5}$;

Multiply these values of the fraction by $H_{\lambda\text{max}} (= 3.42 \times 10^{-5})$ using scale j if desired:

$$H_{\lambda}(0.5) = 1.13 \times 10^{-2} \times 3.42 \times 10^{-5} = 3.865 \times 10^{-7} \text{ watt cm}^{-2} \text{ cm}_{\Delta\lambda}^{-1},$$

$$H_{\lambda}(3) = 0.411 \times 3.42 \times 10^{-5} = 1.409 \times 10^{-5} \text{ watt cm}^{-2} \text{ cm}_{\Delta\lambda}^{-1},$$

$$H_{\lambda}(50) = 2.19 \times 10^{-5} \times 3.42 \times 10^{-5} = 7.5 \text{ watt cm}^{-2} \text{ cm}_{\Delta\lambda}^{-1}.$$

(iii) placing the cursor line over the line $H_{\lambda\text{max}}$ the value of $\lambda''_m = 2.65 \text{ microns}$ may be read from scale e. The various values of H_{ν} may be obtained using the formula:

$$H_{\nu} = (H_{\lambda}/H_{\lambda\text{max}}) \times H_{\lambda\text{max}} \times \lambda_{\mu}^2 \times 10^{-8};$$

$$H_{\nu}(0.5) = 3.865 \times 10^{-7} \times 0.5^2 \times 10^{-8} = 9.66 \text{ watt cm}^{-2} \text{ cm}_{\Delta\nu},$$

$$H_{\nu}(3) = 1.409 \times 10^{-5} \times 3^2 \times 10^{-8} = 1.268 \times 10^{-2} \text{ watt cm}^{-2} \text{ cm}_{\Delta\nu},$$

$$H_{\nu}(50) = 7.5 \times 50^2 \times 10^{-8} = 1.875 \times 10^{-4} \text{ watt cm}^{-2} \text{ cm}_{\Delta\nu}.$$

2. Given a black body at 1000°K , find (i) the total radiant flux density, H , (ii) the radiant flux density $H_{\lambda_1-\lambda_2}$ for the following wavelength intervals: 0.8 to 1.1, 2.5 to 7, and 100 to 300 microns, and (iii) the radiant flux density $H_{0-\lambda_m}$ on the short wavelength side of $H_{\lambda\text{max}}$.

Set the line $H_{\lambda \max}$ on the energy side of the slide against the point $T = 1000^{\circ}\text{K}$ on scale d.

(i) Moving the cursor to the same point, the values $H = 5.71 \text{ watt cm}^{-2}$ and $\lambda_m = 2.90 \text{ microns}$ may be read off from scales a and e.

(ii) Placing the cursor line in turn on the values $\lambda = 0.8, 1.1, 2.5$ and 7 microns (black), read on the black scale m the corresponding values of the fraction $H_{0-\lambda}/H = 1.62 \times 10^{-5}$, 9.2×10^{-4} , 1.65×10^{-1} and 8.09×10^{-1} respectively. In the case of the longer wavelengths of 100 and 300 microns the cursor must be placed on the appropriate values marked in red on scale e, and the values of the fraction $H_{\lambda-\infty}/H$ read off from scale l (red): 1.44×10^{-4} and 5.57×10^{-6} respectively. Taking differences the required energies in the various wavelength intervals are obtained:

$$H(0.8 - 1.1)/H = 9.2 \times 10^{-4} - 1.62 \times 10^{-5} = 9.04 \times 10^{-4} \text{ and}$$

$$H(0.8 - 1.1) = 9.04 \times 10^{-4} \times 5.71 = 5.16 \times 10^{-3} \text{ watt cm}^{-2}.$$

$$H(2.5 - 7)/H = 0.809 - 0.165 = 0.644 \text{ and}$$

$$H(2.5 - 7) = 0.644 \times 5.71 = 3.68 \text{ watt cm}^{-2}.$$

$$H(100 - 300)/H = 1.44 \times 10^{-4} - 5.57 \times 10^{-6} = 1.38 \times 10^{-4} \text{ and}$$

$$H(100 - 300) = 1.38 \times 10^{-4} \times 5.71 = 7.89 \times 10^{-4} \text{ watt cm}^{-2}.$$

(iii) Since the line $H_{\lambda \max}$ on the slide passes through the value $H_{0-\lambda}/H = 0.25$ on scale m, 25% of the energy emitted by the black body is of wavelengths shorter than λ_m . This is a general result and is independent of temperature (2).

$$H_{0-\lambda_m} = 0.25 \times 5.71 = 1.43 \text{ watt cm}^{-2}.$$

3. Find the values λ_m , H , and $H_{\lambda \max}$ for black bodies at temperatures of 0.78°K , 78°K and $78,000^{\circ}\text{K}$.

All these temperatures are beyond the direct range of the slide rule. In accordance with the method described for extension of the temperature scale, a temperature $T_s = 780^{\circ}\text{K}$ is chosen inside the compass of the scale d. In the formula $T_s = T \times 10^n$, $n = 3, 1, -2$ for the three values of temperature. For chosen T_s , $H = 2.08 \text{ watt cm}^{-2}$, $H_{\lambda \max} = 3.75 \times 10^3 \text{ watt cm}^{-2} \text{ cm}_{\Delta\lambda}^{-1}$ and $\lambda_m = 3.71 \text{ microns}$. Then applying the appropriate factors given in section XI and listed in column VIII of the instructions on the back of the rule we find:

For $T = 0.78^{\circ}\text{K}$ ($n = 3$):

$$H = 2.08 \times 10^{-4(+3)} = 2.08 \times 10^{-12} \text{ watt cm}^{-2},$$

$$H_{\lambda \max} = 3.75 \times 10^{-5(+3)} = 3.75 \times 10^{-15} \text{ watt cm}^{-2} \text{ cm}_{\Delta\lambda}^{-1} \text{ and}$$

$$\lambda_m = 3.71 \times 10^3 = 3710 \text{ microns} = 0.371 \text{ cm}.$$

For $T = 78^{\circ}\text{K}$ ($n = 1$):

$$H = 2.08 \times 10^{-4(+1)} = 2.08 \times 10^{-4} \text{ watt cm}^{-2},$$

$$H_{\lambda \max} = 3.75 \times 10^{-5(+1)} = 3.75 \times 10^{-5} \text{ watt cm}^{-2} \text{ cm}_{\Delta\lambda}^{-1} \text{ and}$$

$$\lambda_m = 3.71 \times 10^{+1} = 37.1 \text{ microns}.$$

For/

For $T = 78,000^\circ\text{K}$ ($n = -2$):

$$H = 2.08 \times 10^{-4(-2)} = 2.08 \times 10^8 \text{ watt } \frac{\text{cm}}{\text{cm}^2},$$

$$H_{\lambda\text{max}} = 3.75 \times 10^{-5(-2)} = 3.75 \times 10^{10} \text{ watt } \frac{\text{cm}}{\text{cm}^2} \frac{1}{\text{cm}_{\Delta\lambda}} \text{ and}$$

$$\lambda_m = 3.71 \times 10^{-2} = 0.0371 \text{ microns.}$$

4. Find the maximum photon flux density $Q_{\lambda\text{max}}$, the corresponding wavelength λ'_m and the spectral photon flux densities Q_λ and Q_ν at the wavelengths 0.5, 3 and 50 microns for a black body at a temperature of 1650°C .

Set the line $H_{\lambda\text{max}}$ on the photon side of the slide against the temperature 1650°C on scale g and placing the cursor line on the same point read:

(i) on scale g: $Q_{\lambda\text{max}} = 2.86 \times 10^{24} \text{ photon sec}^{-1} \frac{\text{cm}}{\text{cm}^2} \frac{1}{\text{cm}_{\Delta\lambda}};$

(ii) move the cursor line to the point marked $Q_{\lambda\text{max}}$ on scale n and read on scale e $\lambda'_m = 1.91 \text{ microns};$

(iii) moving the cursor line in turn to the values 0.5, 3, and 50 microns on scale e (the value for the latter wavelength being found in red), read the values of the fraction $Q_\lambda/Q_{\lambda\text{max}}$ from scale n for the two shorter wavelengths and from scale o (red) for the longer wavelength:

for $\lambda = 0.5$; $Q_\lambda/Q_{\lambda\text{max}} = 3.3 \times 10^{-3},$

for $\lambda = 3$; $Q_\lambda/Q_{\lambda\text{max}} = 0.731,$

for $\lambda = 50$; $Q_\lambda/Q_{\lambda\text{max}} = 6.55 \times 10^{-4}.$

Multiplying these values by $Q_{\lambda\text{max}}$ (using scale p if desired) the required spectral photon flux densities are obtained:

$$Q_\lambda(0.5) = 3.3 \times 10^{-3} \times 2.86 \times 10^{24} = 9.44 \times 10^{21} \text{ photon sec}^{-1} \frac{\text{cm}}{\text{cm}^2} \frac{1}{\text{cm}_{\Delta\lambda}},$$

$$Q_\lambda(3) = 0.731 \times 2.86 \times 10^{24} = 2.09 \times 10^{24} \text{ photon sec}^{-1} \frac{\text{cm}}{\text{cm}^2} \frac{1}{\text{cm}_{\Delta\lambda}},$$

$$Q_\lambda(50) = 6.55 \times 10^{-4} \times 2.86 \times 10^{24} = 1.871 \times 10^{21} \text{ photon sec}^{-1} \frac{\text{cm}}{\text{cm}^2} \frac{1}{\text{cm}_{\Delta\lambda}}.$$

(iv) placing the cursor line over the point $Q_{\nu\text{max}}, \lambda'''_m = 4.7 \text{ microns}$ may be read from scale e. The appropriate values of Q_ν may be obtained using the formula:

$$Q_\nu = (Q_\lambda/Q_{\lambda\text{max}}) \times Q_{\lambda\text{max}} \times \lambda_\mu^2 \times 10^{-8};$$

$$Q_\nu(0.5) = 9.44 \times 10^{21} \times 0.5^2 \times 10^{-8} = 2.36 \times 10^{13} \text{ photon sec}^{-1} \frac{\text{cm}}{\text{cm}^2} \frac{1}{\text{cm}_{\Delta\nu}},$$

$$Q_\nu(3) = 2.09 \times 10^{24} \times 3^2 \times 10^{-8} = 1.881 \times 10^{17} \text{ photon sec}^{-1} \frac{\text{cm}}{\text{cm}^2} \frac{1}{\text{cm}_{\Delta\nu}},$$

$$Q_\nu(50) = 1.871 \times 10^{21} \times 50^2 \times 10^{-8} = 4.68 \times 10^{16} \text{ photon sec}^{-1} \frac{\text{cm}}{\text{cm}^2} \frac{1}{\text{cm}_{\Delta\nu}}.$$

5. Find the photon flux densities for a black body at a temperature of 1000°K in the wavelength intervals 0.8 to 1.1, 2.5 to 7, and 100 to 300 microns.

Set the line $H_{\lambda\text{max}}$ on the photon side of the slide against the point $T = 1000^\circ\text{K}$ on scale d, and placing the cursor line on the same point, read from scale f the value $Q = 1.52 \times 10^{20} \text{ photon sec}^{-1} \frac{\text{cm}}{\text{cm}^2}$. Placing the cursor line in turn on the points $\lambda = 0.8, 1.1, 2.5,$ and 7 microns on scale e read the corresponding values of the fraction

$$Q_{0-\lambda}/Q = /$$

$Q_{0-\lambda}/Q = 2.3 \times 10^{-6}$, 1.7×10^{-4} , 6.15×10^{-2} and 0.575 respectively on scale g. The values appropriate to the longer (red) wavelengths are read from scale r (red): $Q_{\lambda-\infty}/Q = 4.11 \times 10^{-3}$ and 4.71×10^{-4} for $\lambda = 100$ and 300 microns respectively. Taking differences, the required values of the fraction for the three wavelength intervals are obtained and hence the values of $Q_{\lambda_1-\lambda_2}$:

$$\begin{aligned} Q(0.8 - 1.1)/Q &= 1.7 \times 10^{-4} - 2.3 \times 10^{-6} = 1.68 \times 10^{-4} \text{ and} \\ Q(0.8 - 1.1) &= 1.68 \times 10^{-4} \times 1.52 \times 10^{20} = 2.55 \times 10^{16} \text{ photon sec}^{-1} \text{ cm}^{-2}. \\ Q(2.5 - 7)/Q &= 0.575 - 6.15 \times 10^{-2} = 0.513 \text{ and} \\ Q(2.5 - 7) &= 0.513 \times 1.52 \times 10^{20} = 7.8 \times 10^{19} \text{ photon sec}^{-1} \text{ cm}^{-2}. \\ Q(100 - 300)/Q &= 4.11 \times 10^{-3} - 4.71 \times 10^{-4} = 3.64 \times 10^{-3} \text{ and} \\ Q(100 - 300) &= 3.64 \times 10^{-3} \times 1.52 \times 10^{20} = 5.54 \times 10^{17} \text{ photon sec}^{-1} \text{ cm}^{-2}. \end{aligned}$$

6. Given a black body normally at a temperature of 800°C and a receiver with sensitivity in the wavelength interval 1.2 to 1.5 microns, calculate the variation in receiver output if the black body temperature varies from 600°C to 1000°C.

It is necessary to consider the variations in $H(1.2 - 1.5)$ resulting from the variations in source temperature. The variation of H with temperature must first be determined. Taking $H(800) = 1$, place the point "10⁻¹" of multiplication scale k against $t = 800^\circ\text{C}$ on scale g. Any unity point of scale k may, of course, be chosen that allows all the necessary operations to be carried out on both sides of the unity point selected. Moving the cursor line in turn to the various temperature values listed in the table below (column 1) the values of the ratio $H(t)/H(800)$, given in column (2) of the table, may be read from scale k. By placing the line $H_{\lambda\text{max}}$ in turn on the different temperatures, the corresponding values of $H(0 - 1.2)/H$ and $H(0 - 1.5)/H$ may be read from scale m, and the differences taken (columns 3, 4 and 5). Multiplying the figures in column (5) by those in column (2), the relative values for the energy received at each temperature are obtained (column 6). These are converted to fractions of $H_{\Delta\lambda}(800)$ in column (7) by dividing by 1.66×10^{-2} . The results are presented graphically in Figure 3. In this case the values of the fraction for $t < 800^\circ\text{C}$ have also been plotted in the form $H_{\Delta\lambda}(800)/H_{\Delta\lambda}(t)$ to increase the accuracy of reading at the lower temperatures.

TABLE./

t (1)	H(t)/H(800) (2)	H(0 - 1.5)/H (3)	H(0 - 1.2)/H (4)	(3) - (4) (5)	(5) x (2) (6)	H _{Δλ} (t)/H _{Δλ} (800) (7)
600	0.438	4.60 x 10 ⁻³	5.40 x 10 ⁻⁴	4.06 x 10 ⁻³	1.77 x 10 ⁻³	0.1067
625	0.491	5.80	7.50	5.05	2.48	0.1492
650	0.548	7.20	1.00 x 10 ⁻³	6.20	3.40	0.205
675	0.609	8.80	1.29	7.51	4.57	0.276
700	0.676	1.06 x 10 ⁻²	1.62	8.98	6.07	0.365
725	0.748	1.27	2.05	1.065 x 10 ⁻²	7.96	0.48
750	0.826	1.50	2.60	1.24	1.025 x 10 ⁻²	0.617
775	0.910	1.76	3.20	1.44	1.31	0.79
800	1.000	2.06	4.00	1.66	1.66	1
825	1.097	2.37	4.80	1.885	2.07	1.245
850	1.200	2.68	5.70	2.11	2.53	1.522
875	1.310	3.01	6.75	2.335	3.06	1.841
900	1.428	3.38	8.00	2.58	3.68	2.215
925	1.554	3.78	9.40	2.84	4.41	2.66
950	1.688	4.23	1.10 x 10 ⁻²	3.13	5.28	3.18
975	1.830	4.72	1.26	3.46	6.33	3.81
1000	1.981	5.25	1.43	3.82	7.60	4.58

7. Find the energy emitted by a black body at temperature 200°C in a 10 Mcycle bandwidth centred about 10 centimetres wavelength.

A wavelength of 10 centimetres corresponds to a frequency of 3×10^9 cycles sec^{-1} . For a bandwidth of 10 Mcycle sec^{-1} , ($= \frac{1}{3} \times 10^{-3} \text{ cm}_{\Delta\nu}^{-1} = \Delta\nu$), the extreme frequencies are $3 \times 10^9 \pm 0.5 \times 10^7$ cycle sec^{-1} . The narrowness of the bandwidth renders calculations involving $(H_{\lambda_1-\infty}/H) - (H_{\lambda_2-\infty}/H)$ inaccurate, but in such a case it may be assumed that in the given interval H_ν is constant, and then the value calculated for $\lambda = 10^5$ microns, and multiplied by the factor $\Delta\nu$. Since this wavelength is beyond the range of the slide rule, it is necessary to choose a value of wavelength $\lambda_s = \lambda \times 10^{-n}$ where n is the smallest integer that will bring λ_s inside the normal red range of the rule. In the present case $n = 2$ and $\lambda_s = 10^5 \times 10^{-2} = 1000$ microns. Set the line $H_{\lambda\text{max}}$ on the temperature 200°C on scale g, and read on scale i (red) the value of $H_\lambda/H_{\lambda\text{max}} = 3.98 \times 10^{-8}$ corresponding to $\lambda_s = 1000$ microns. For 200°C, $H_{\lambda\text{max}} = 3.1 \times 10^2 \text{ watt cm}^{-2} \text{ cm}_{\Delta\lambda}^{-1}$ and therefore applying the formula

$$H_\nu = (H_\lambda/H_{\lambda\text{max}}) \times H_{\lambda\text{max}} \times \lambda_\mu^2 \times 10^{-8}$$

$$\text{we have } H_\nu(1000) = 3.98 \times 10^{-8} \times 3.1 \times 10^2 \times 10^6 \times 10^{-8}$$

$$= 1.235 \times 10^{-7} \text{ watt cm}^{-2} \text{ cm}_{\Delta\nu}$$

Converting/

Converting this to the original wavelength: $H_\nu(10 \text{ cm}) = 1.235 \times 10^{-7} \times 10^{-3(+2)} = 1.235 \times 10^{-13} \text{ watt cm}^{-2} \text{ cm}_{\Delta\nu}$ and for interval $\Delta\nu = \frac{1}{3} \times 10^{-3} \text{ cm}_{\Delta\nu}^{-1}$ we get

$$H_{\Delta\nu}(10 \text{ cm}) = 1.235 \times 10^{-13} \times \frac{1}{3} \times 10^{-3} = 4.12 \times 10^{-17} \text{ watt cm}^{-2}.$$

XIV. COMPARISON OF THE PLANCKIAN FORMULA WITH WIEN AND RAYLEIGH-JEANS FORMULAE.

Both the Wien and Rayleigh-Jeans formulae may be treated as mathematical approximations of the exact Planck formula. The latter may be written in the form:

$$(H_\lambda)_p T^{-5} = \frac{c_1}{c_2^5} \frac{x^5}{e^x - 1}.$$

It is possible to deduce either the Wien formula or the Rayleigh-Jeans formula and obtain the errors introduced by the use of these formulae in terms of x or λT .

If $x \gg 1$ (λT very small) then e^x is sufficiently larger than unity for $e^x - 1$ to be replaced by e^x , when:

$$(H_\lambda)_w T^{-5} = \frac{c_1}{c_2^5} \frac{x^5}{e^x}$$

This is Wien's formula which is usually written in the form:

$$(H_\lambda)_w = c_1 \lambda^{-5} e^{-c_2/\lambda T}$$

The error involved in using this approximation is:

$$E_w = \frac{(H_\lambda)_w - (H_\lambda)_p}{(H_\lambda)_p} = \frac{x^5 e^{-x} - x^5/(e^x - 1)}{x^5/(e^x - 1)} = e^{-x}(e^x - 1) - 1 = -e^{-x} = -e^{-c_2/\lambda T}$$

If, on the other hand, $x \ll 1$ (λT very large) then:

$$e^x - 1 = (1 + \frac{x}{1} + \frac{x^2}{1.2} + \dots) - 1 = x + \frac{x^2}{2} \dots \approx x$$

$$\text{and } (H_\lambda)_r T^{-5} = \frac{c_1}{c_2^5} x^4$$

This is, of course, the Rayleigh-Jeans formula and is usually written in the form:

$$(H_\lambda)_r = \frac{c_1}{c_2} T \lambda^{-4}$$

The relative error is:

$$E_r = \frac{(H_\lambda)_r - (H_\lambda)_p}{(H_\lambda)_p} = \frac{x^4 - x^5/(e^x - 1)}{x^5/(e^x - 1)} = \frac{e^x - 1 - x}{x} = \frac{x}{2} + \frac{x^2}{6} + \dots = \frac{1}{2} c_2/\lambda T + \frac{1}{6} (c_2/\lambda T)^2 + \dots$$

The corresponding simplified formulae for H_ν , Q_λ and Q_ν introduce similar errors as a function of x .

In/

In the case of calculations involving $H_{\lambda_1-\lambda_2}$ or $Q_{\lambda_1-\lambda_2}$ estimation of the error is more complicated. The value will depend upon the values of both x_1 and x_2 corresponding to λ_1 and λ_2 . It is possible nevertheless to indicate the largest possible value of error introduced by using either formula.

In the case of the Wien formula:

$$E_w = \frac{(H_{\lambda_1-\lambda_2})_w - (H_{\lambda_1-\lambda_2})_p}{(H_{\lambda_1-\lambda_2})_p} = \frac{\int_{x_1}^{x_2} \frac{-x^3 dx}{e^x} - \int_{x_1}^{x_2} \frac{-x^3 dx}{e^x - 1}}{\int_{x_1}^{x_2} \frac{-x^3 dx}{e^x - 1}} = \frac{\int_{x_1}^{x_2} (-e^{-x}) \cdot \frac{-x^3 dx}{e^x - 1}}{\int_{x_1}^{x_2} \frac{-x^3 dx}{e^x - 1}}$$

The integrands in the numerator and denominator differ by the variable factor $-e^{-x}$. However, this may be replaced by a constant factor $-e^{-x_1}$ (where x_1 has some value intermediate between x_1 and x_2), that can be removed from under the integral sign without altering the result, when:

$$E_w = -e^{-x_1}$$

The value of x_1 is not known, but if x_1 is replaced by x_1 (for $x_1 < x_2$), it can be stated that the absolute value of the error certainly will be less than e^{-x_1} :

$$E_w \leq e^{-x_1} = e^{-c_2/\lambda_1 T} \quad (x_1 < x_2)$$

This result is, of course, similar to that obtained for the simpler quantities.

Similarly for the Rayleigh-Jeans formula:

$$E_r = \frac{(H_{\lambda_1-\lambda_2})_r - (H_{\lambda_1-\lambda_2})_p}{(H_{\lambda_1-\lambda_2})_p} = \frac{\int_{x_1}^{x_2} (\mathcal{L}) \cdot \frac{-x^3 dx}{e^x - 1}}{\int_{x_1}^{x_2} \frac{-x^3 dx}{e^x - 1}} = \mathcal{L}_1$$

where $\mathcal{L} = \frac{1}{2}x + \frac{1}{6}x^2 + \dots$ and $\mathcal{L}_1 = \frac{1}{2}x_1 + \frac{1}{6}x_1^2 + \dots$ ($x_1 < x_1 < x_2$)

As before, replacing x_1 by x_2 (for $x_1 < x_2$), the maximum possible value of error is obtained:

$$E_r \leq \frac{1}{2}x_2 + \frac{1}{6}x_2^2 + \dots = \frac{1}{2}c_2/\lambda_2 T + \frac{1}{6}(c_2/\lambda_2 T)^2 + \dots \quad (x_1 < x_2)$$

The corresponding formulae for $Q_{\lambda_1-\lambda_2}$ introduce similar errors.

Figure 4 represents errors for both of the simplified formulae as a function of T . The errors introduced by using either formula at given values of λ or λT can be estimated using this graph.

ACKNOWLEDGMENT.

It is desired to acknowledge the assistance of Mathematics Division, N.P.L. in checking the calculations involved in the preparation of certain of the scales of the slide rule.

M. Makowski, T.E.A.II.

L.A.J. Verra, A.E.O.

Mr. LAJV/VBC.

8.9.47.

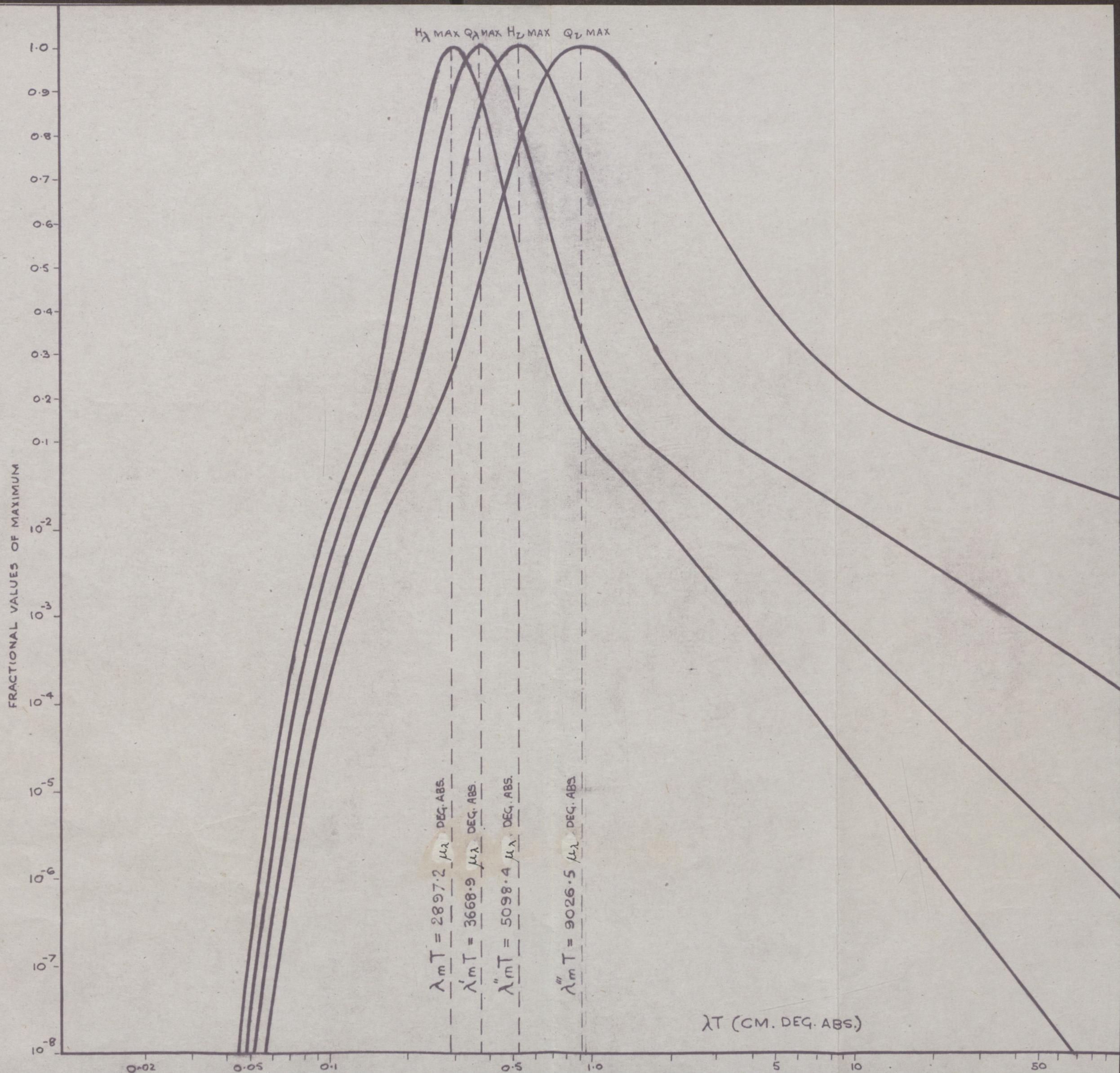
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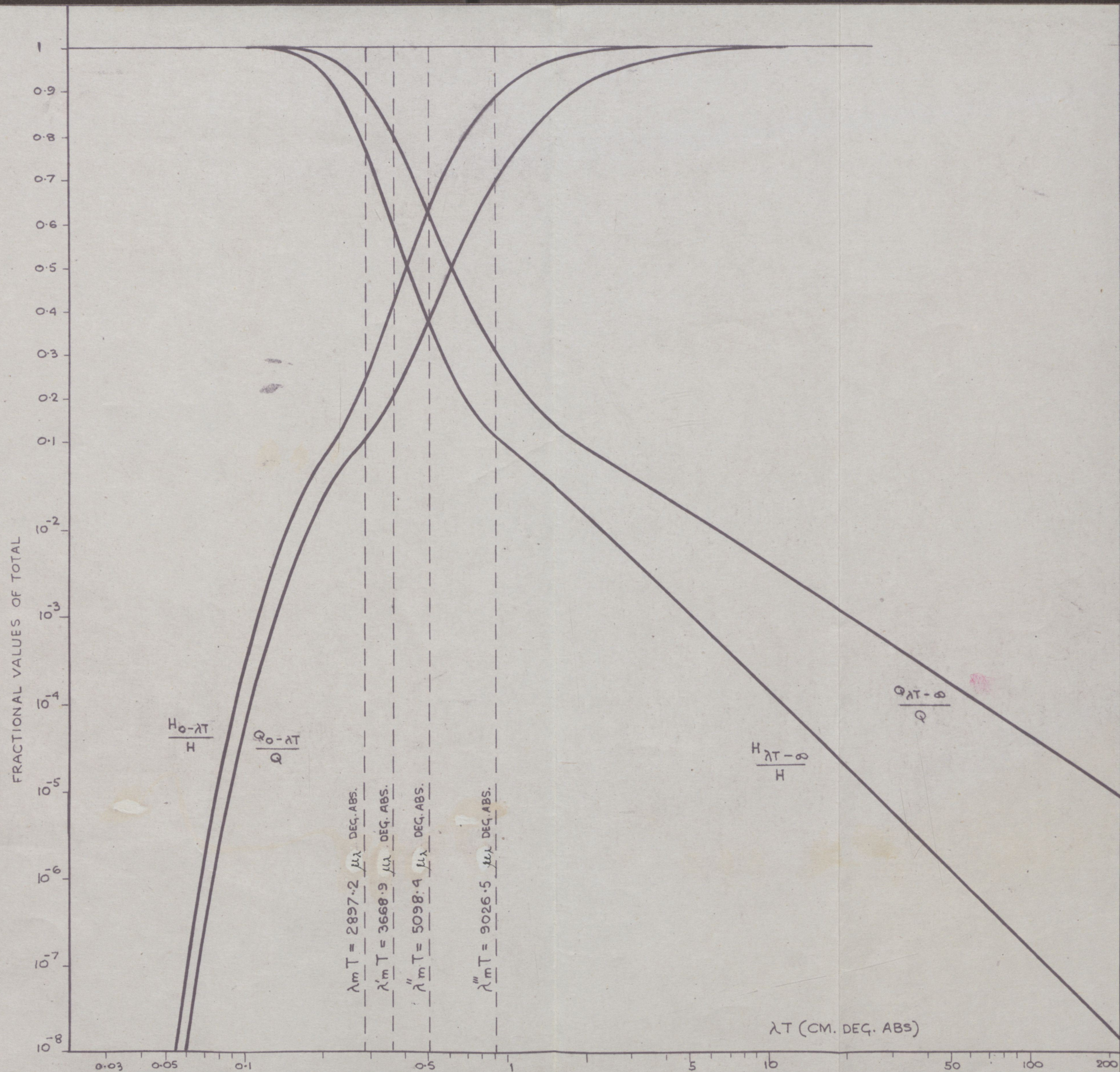
SPECTRAL CURVES.

FIG 1

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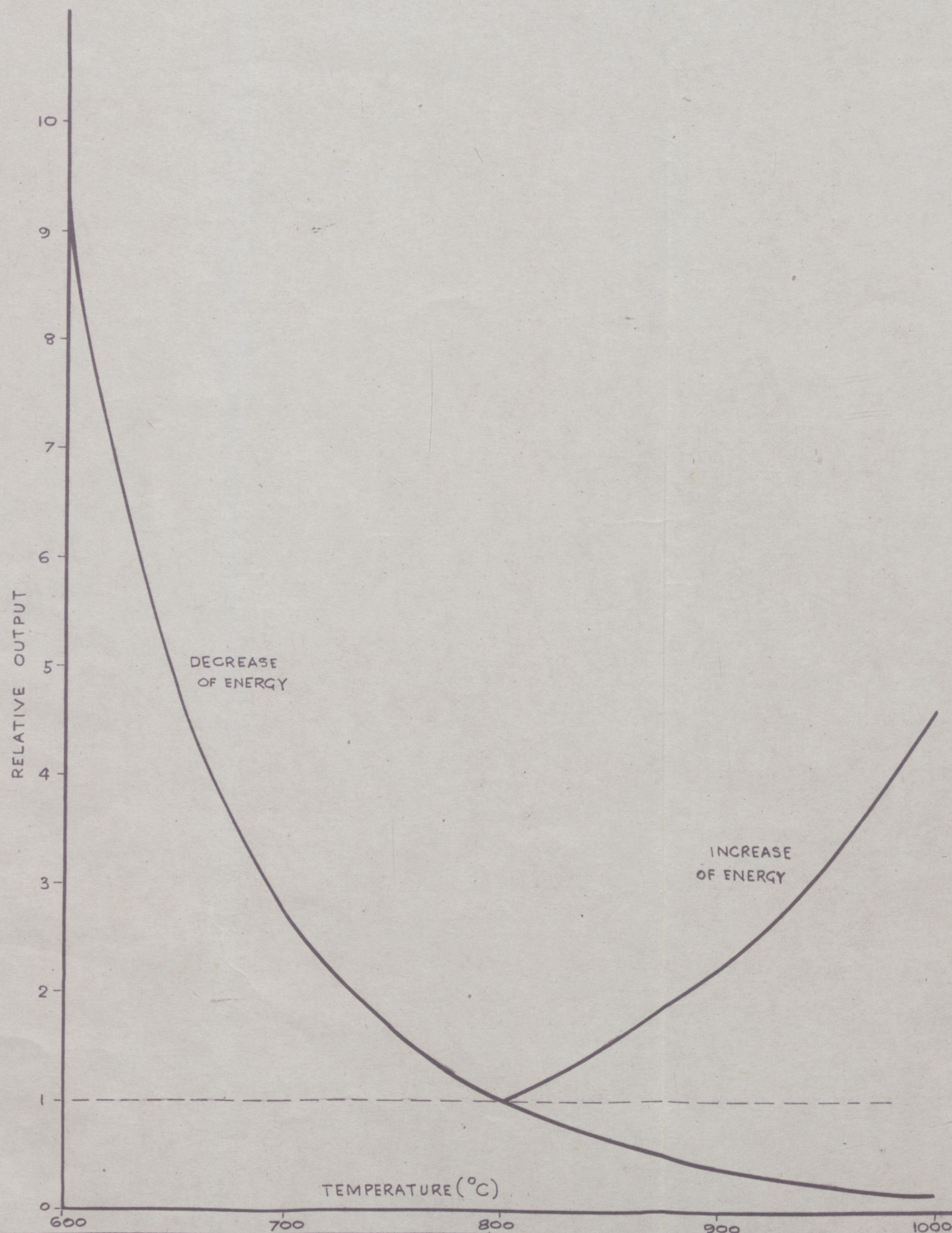
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FIG 2

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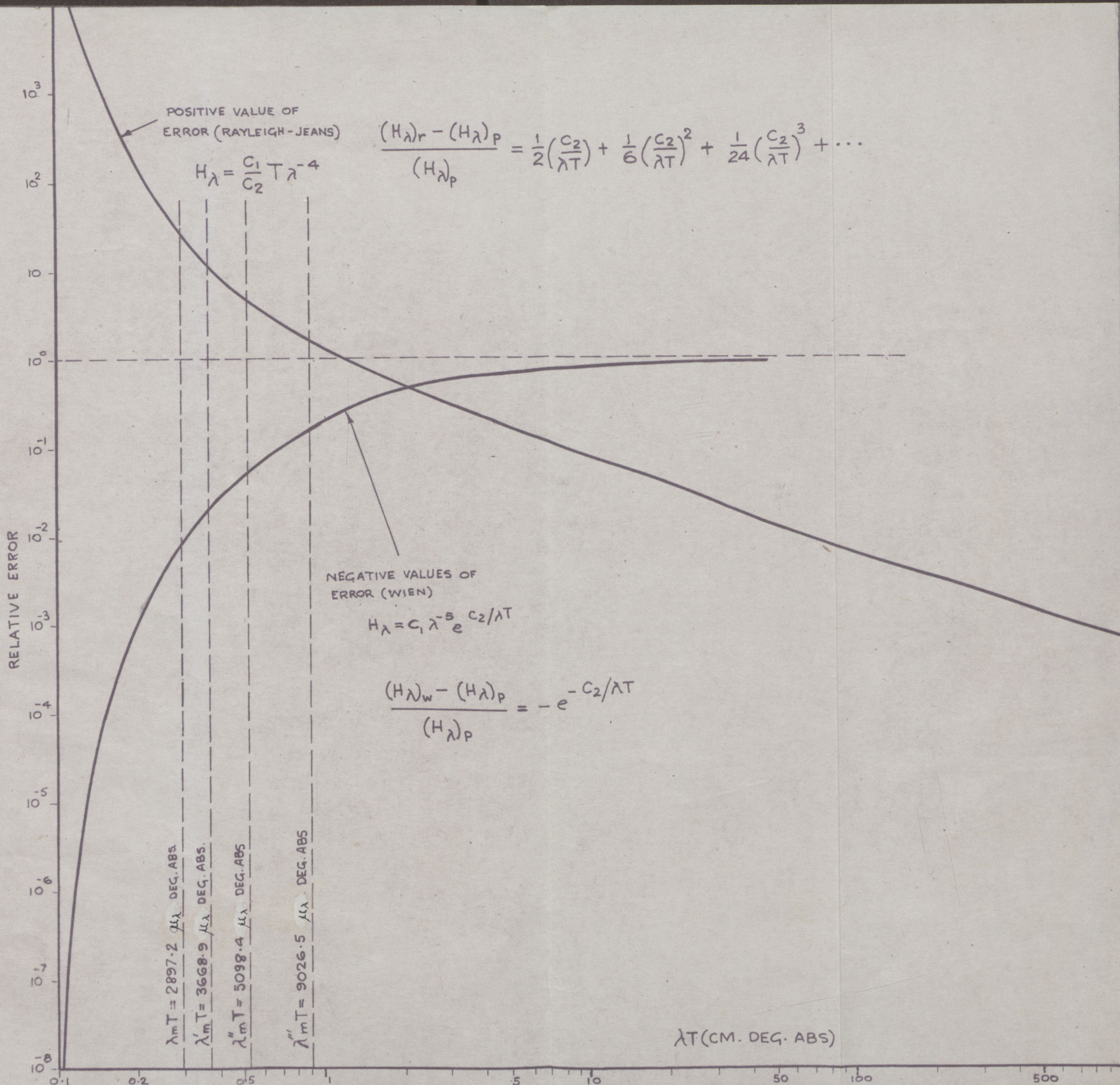
TITLE VARIATION OF OUTPUT WITH TEMPERATURE FOR A SOURCE NORMALLY AT 800°C. (SEE EXAMPLE 6)

FIG 3

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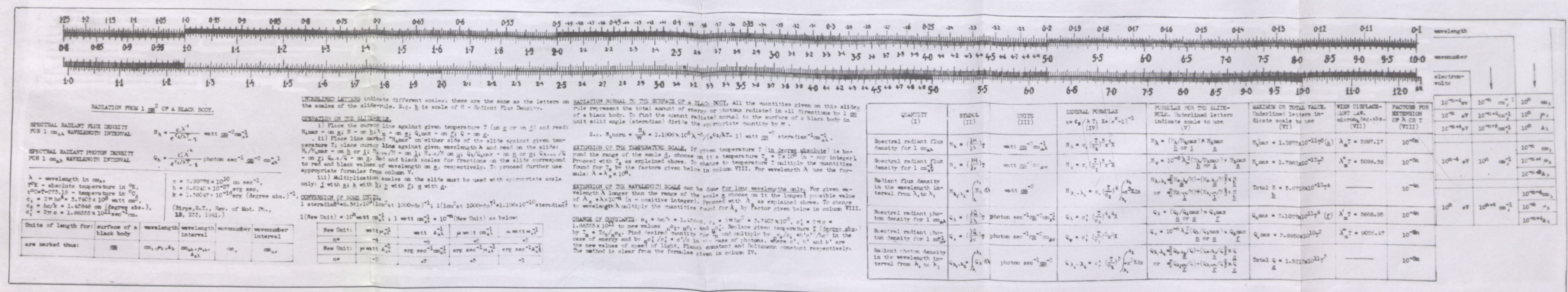
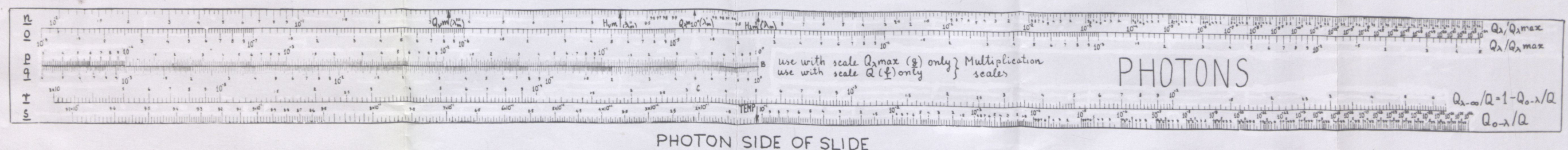
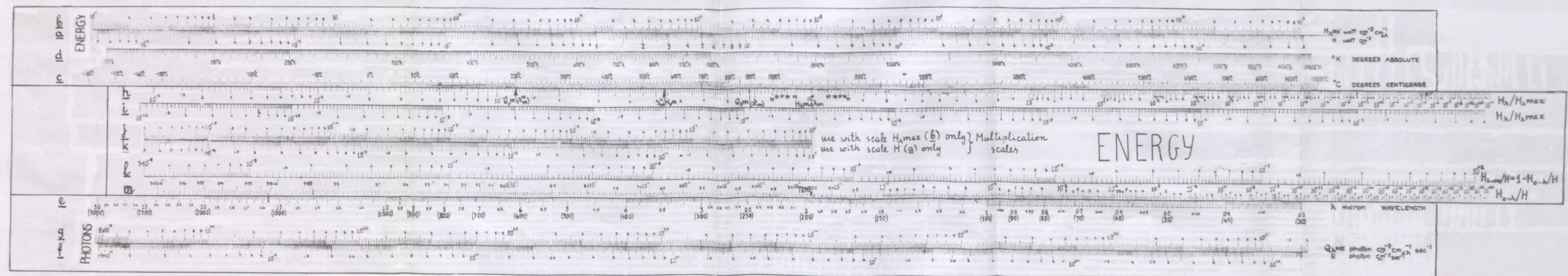
TITLE ERROR INTRODUCED BY APPLICATION OF WIEN OR RAYLEIGH-JEANS FORMULAE.

FIG 4

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SCALES AND BRIEF INSTRUCTIONS FOR SLIDE RULE

FIG. 5

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